

Minimum uncertainty state and non-negative quasi probability of qudit

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For a qudit, a quantum system on a finite dimensional Hilbert space, we construct minimum uncertainty states and a non-negative quasi probability distribution. They are the discrete version of the coherent states and the Husimi function in the continuous quantum mechanics. The two bases related by the discrete Fourier transformation are identified with the position and momentum bases. We prove an inequality concerning uncertainties of the position and momentum of a qudit. The minimum uncertainty states are the ones which saturate the inequality. Like the coherent states, they are not mutually orthogonal but comprise an overcomplete set in the state space. As the dimension of the Hilbert space goes to infinity, these states approach the coherent states. Using the minimum uncertain states, we construct a non-negative quasi probability distribution for a qudit. Its marginal distributions are smeared out, but we show that there does not exist a non-negative quasi probability distribution with sharper marginal properties if the translational covariance is assumed.

I. INTRODUCTION

The uncertainty principle [1] is arguably one of the most fundamental features which differentiates quantum mechanics from classical one. It states that the product of uncertainties in complementary physical observables (*e.g.* position and momentum) has an inherent finite lower bound, and it has a profound influence on our view of the physical world. Because of the uncertainty principle, dynamics of a quantum system is qualitatively different from classical one; for example an atom would collapses without this principle. Furthermore recent studies show that the uncertainty principle also plays an important role in a variety of quantum information processing [2]. For example, the quantum cryptography [3], one of the most remarkable application of quantum information, exploits the uncertainty principle together with the no-cloning theorem [4] to ensure its provable security.

The uncertainty relation of the position and momentum in the continuous quantum mechanics is expressed by an inequality involving the standard deviations of their distributions [5]; that is, $\Delta x \Delta p \geq 1/2$. The states which attain the minimum are called minimum uncertainty states, and given by the coherent states. The coherent states, the eigenstates of the annihilation operator, have interesting properties and useful applications in various fields of physics (see *e.g.* Ref. [6]). Using the coherent states, one can define a quasi probability distribution for the position and momentum variables, which is called the Husimi function [7]. The Husimi function is always non-negative in contrast to the Wigner function [8], which is another quasi distribution function and may take negative values except for the case of Gaussian wave functions [9].

In this paper we present a finite-dimensional version of the minimum uncertainty states of a qudit, a quantum system on a finite dimensional Hilbert space, and provide a non-negative quasi probability distribution based on these states.

For this purpose we first need an appropriate measure of uncertainty of the position and momentum variables

of a qudit. We show that the modulus of the expectation value of the position (momentum) translation operator is suitable for quantifying the uncertainty of the position (momentum) distribution (Sec. II). We then prove an inequality involving the product of these two moduli corresponding to the position and momentum uncertainty (Sec. III). We call the states saturating this inequality minimum uncertainty states, which comprise an overcomplete set in the state space. As expected, it can be shown that these minimum uncertainty states approach the coherent states as the dimension of the state space goes to infinity (Sec. IV).

In the same way as in the continuous quantum mechanics, we define a quasi probability distribution of a qudit using the minimum uncertainty states (Sec. V). This is a finite-dimensional version of the Husimi function, and non-negative at the cost of the smeared out marginal distributions. We show that the obtained quasi probability distribution is optimal in the sense that there exists no non-negative quasi probability distribution with sharper marginal properties if the translational covariance is assumed (Sec. VI).

II. POSITION AND MOMENTUM UNCERTAINTY OF A QUDIT

We consider a qudit, a quantum system described by a d -dimensional complex linear space \mathbb{C}^d . An orthonormal basis $\{|a\rangle\}_{a=0}^{d-1}$ is fixed to define the “position” coordinate a . We introduce another orthonormal basis, which is the discrete Fourier transform defined by

$$|\tilde{b}\rangle = \frac{1}{\sqrt{d}} \sum_{a=0}^{d-1} \omega^{ba} |a\rangle, \quad b = 0, 1, \dots, d-1, \quad (1)$$

where $\omega = e^{2\pi i/d}$ is a primitive d th root of unity. The index b is interpreted as the “momentum” coordinate. These two bases are unbiased in the sense that $|\langle a|\tilde{b}\rangle| = 1/\sqrt{d}$ for all a and b , and expected to approach the continuous position and momentum bases as the dimension

d goes to infinity. As a feature of the discrete Fourier transform, the position and momentum coordinates, a and b , can not simultaneously have sharp values.

In order to quantify the uncertainty with respect to these two bases, we employ two unitary operators Q and P . The operator Q is given by

$$Q = \sum_{a=0}^{d-1} \omega^a |a\rangle \langle a|, \quad (2)$$

which is diagonal in the position basis $\{|a\rangle\}$. In the momentum basis $\{|\tilde{b}\rangle\}$, the operator Q translationally shifts the momentum coordinate as $Q|\tilde{b}\rangle = |\widetilde{b+1}\rangle$. Here, it is assumed that if $b+1 = d$ then $|\widetilde{b+1}\rangle$ is equal to $|\tilde{0}\rangle$. Throughout this paper we employ this periodic convention for the position and momentum coordinates; namely, we assume that

$$|a+d\rangle = |a\rangle, \quad |\widetilde{b+d}\rangle = |\tilde{b}\rangle, \quad (3)$$

for any integers a and b . Another operator P is defined by

$$P = \sum_{b=0}^{d-1} \omega^{-b} |\tilde{b}\rangle \langle \tilde{b}|, \quad (4)$$

which is diagonal in the momentum basis, and in the position basis it acts as the translational operator; $P|a\rangle = |a+1\rangle$. It is readily shown that P and Q satisfy the following relations:

$$Q^d = P^d = \mathbf{1}, \quad QP = \omega PQ. \quad (5)$$

The relation $QP = \omega PQ$ can be regarded as the counterpart of the canonical commutation relation of the continuous position and momentum operators.

For a general state $|\phi\rangle$, we write

$$|\phi\rangle = \sum_{a=0}^{d-1} c_a |a\rangle = \sum_{b=0}^{d-1} \tilde{c}_b |\tilde{b}\rangle, \quad (6)$$

where c_a and \tilde{c}_b are expansion coefficients in the position and momentum basis, respectively. Then the expectation values of Q and P for the state $|\phi\rangle$ take the following form:

$$\langle \phi|Q|\phi\rangle = \sum_{a=0}^{d-1} |c_a|^2 \omega^a = \sum_{b=0}^{d-1} \tilde{c}_{b+1}^* \tilde{c}_b, \quad (7)$$

$$\langle \phi|P|\phi\rangle = \sum_{a=0}^{d-1} c_{a+1}^* c_a = \sum_{b=0}^{d-1} |\tilde{c}_b|^2 \omega^{-b}. \quad (8)$$

Now let us examine the expectation value $\langle \phi|Q|\phi\rangle$ expressed in terms of c_a . This is an average of roots of unity ω^a with weights given by $|c_a|^2$. In the complex plane the points $\{\omega^a\}_{a=0}^{d-1}$ are at the vertices of a regular d -sided polygon inscribed in the unit circle, and the expectation value $\langle \phi|Q|\phi\rangle$ is somewhere in this polygon (see

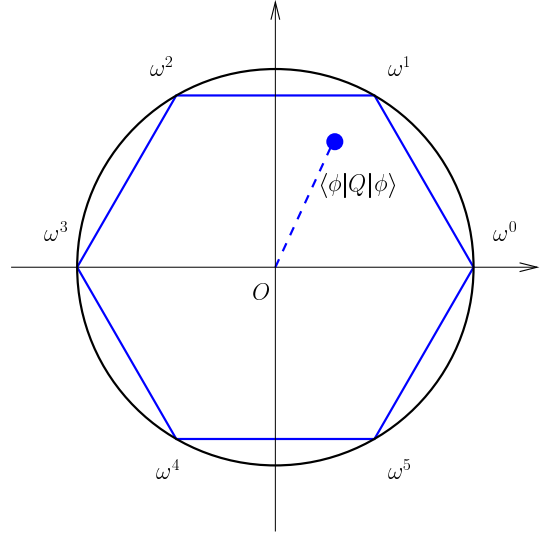


FIG. 1. (Color online) The d th roots of unity in the complex plane and the expectation value $\langle \phi|Q|\phi\rangle$ represented by a point in the regular d -sided polygon formed by these roots. This figure displays the case of $d = 6$.

Fig. 1). If the position coordinate has a sharp value, say a_0 , $\langle \phi|Q|\phi\rangle$ is at the vertex ω^{a_0} . In this case, and only in this case, $|\langle \phi|Q|\phi\rangle|$ is equal to 1, otherwise we generally have $|\langle \phi|Q|\phi\rangle| < 1$. In contrast, if the weight is equally distributed as $|c_a|^2 = 1/d$, $\langle \phi|Q|\phi\rangle$ is at the origin; that is, $|\langle \phi|Q|\phi\rangle| = 0$. Thus the quantity $|\langle \phi|Q|\phi\rangle|$ is a measure of quantifying how sharply the position coordinate is distributed. In the same way the quantity $|\langle \phi|P|\phi\rangle|$ measures the sharpness of the distribution of momentum coordinate. However, the quantities $|\langle \phi|Q|\phi\rangle|$ and $|\langle \phi|P|\phi\rangle|$ cannot simultaneously have their maximum value 1. For example, take the case of $|\langle \phi|Q|\phi\rangle| = 1$ which occurs only when $|c_a|$ is nonzero for a certain single value of a . In this case, however, $|\langle \phi|P|\phi\rangle|$ must be 0 as its expression in terms of c_a clearly shows.

Motivated by these considerations, we define the certainty C of a state $|\phi\rangle$ to be

$$C(|\phi\rangle) = |\langle \phi|Q|\phi\rangle \langle \phi|P|\phi\rangle|, \quad (9)$$

to quantify the mutual uncertainty with respect to the position and momentum coordinates. Note that a larger C means less uncertainty as the name ‘‘certainty’’ indicates.

III. MINIMUM UNCERTAINTY STATES

In the preceding section, we have seen that the certainty $C(|\phi\rangle)$ in Eq. (9) serves as a measure of certainty of position and momentum for a qudit state $|\phi\rangle$. In this section we study the maximum value of the certainty and the states attaining the maximum certainty, that is, the states with the minimum uncertainty.

Let us first examine the case of a qubit, $d = 2$. In the 2-dimensional case, the operators Q and P are given by the Pauli matrices,

$$\begin{aligned} Q &= |0\rangle\langle 0| - |1\rangle\langle 1| = \sigma_z, \\ P &= |0\rangle\langle 1| + |1\rangle\langle 0| = \sigma_x. \end{aligned}$$

The states are conveniently expressed by the Bloch vector representation.

$$|\mathbf{n}\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\varphi} \sin \frac{\theta}{2} |1\rangle, \quad (10)$$

where $\mathbf{n} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$ is the Bloch vector. For the certainty C of the state $|\mathbf{n}\rangle$, we obtain

$$C(\mathbf{n}) = |\langle \mathbf{n} | \sigma_z | \mathbf{n} \rangle \langle \mathbf{n} | \sigma_x | \mathbf{n} \rangle| = |n_z n_x|. \quad (11)$$

The upper bound of $C(\mathbf{n})$ is readily determined by using the following inequalities:

$$|n_z n_x| \leq \frac{n_x^2 + n_z^2}{2} \leq \frac{1}{2}. \quad (12)$$

Thus the maximum value of the certainty C is $1/2$, and the maximum is attained by the following four Bloch vectors:

$$\mathbf{n}^{(\alpha, \beta)} = \frac{1}{\sqrt{2}} ((-1)^\alpha, 0, (-1)^\beta), \quad (\alpha, \beta = 0, 1). \quad (13)$$

The state with $\mathbf{n}^{(0,0)}$ is denoted $|\Gamma\rangle$, and takes the following explicit form:

$$\begin{aligned} |\Gamma\rangle &\equiv \cos \frac{\pi}{8} |0\rangle + \sin \frac{\pi}{8} |1\rangle \\ &= \frac{\sqrt{2+\sqrt{2}}}{2} |0\rangle + \frac{\sqrt{2-\sqrt{2}}}{2} |1\rangle. \end{aligned} \quad (14)$$

It should be noticed that the four states attaining the maximum C can be expressed as

$$|\alpha, \beta\rangle \equiv \sigma_x^\alpha \sigma_z^\beta |\Gamma\rangle, \quad (\alpha, \beta = 0, 1). \quad (15)$$

Now we generalize these results to arbitrary dimensional cases, and present our main results on the upper bound of the certainty C in general dimensions.

Theorem: For any normalized state $|\phi\rangle$,

(i) The certainty C is bounded by the inequality,

$$C(|\phi\rangle) \equiv |\langle \phi | Q | \phi \rangle \langle \phi | P | \phi \rangle| \leq h^2, \quad (16)$$

where h is the greatest eigenvalue of Hermitian operator H given by

$$H \equiv (P + P^\dagger + Q + Q^\dagger)/4. \quad (17)$$

(ii) Equality in (16) holds if and only if

$$|\phi\rangle = P^\alpha Q^\beta |\Gamma\rangle \quad (\text{up to a global phase}), \quad (18)$$

where $|\Gamma\rangle$ is the non-degenerate eigenstate of H with the greatest eigenvalue h , and α and β are integers $(\alpha, \beta = 0, 1, \dots, d-1)$.

The states $|\alpha, \beta\rangle \equiv P^\alpha Q^\beta |\Gamma\rangle$ are called the minimum uncertainty states.

A. Proof of statement (i) in Theorem

In order to prove statement (i) in Theorem, we start with an inequality

$$\sqrt{|\langle \phi | Q | \phi \rangle \langle \phi | P | \phi \rangle|} \leq \frac{1}{2} (|\langle \phi | Q | \phi \rangle| + |\langle \phi | P | \phi \rangle|), \quad (19)$$

where equality holds if and only if $|\langle \phi | Q | \phi \rangle| = |\langle \phi | P | \phi \rangle|$.

We write a given state $|\phi\rangle$ in the basis $\{|a\rangle\}$ as

$$|\phi\rangle = \sum_{a=0}^{d-1} c_a |a\rangle. \quad (20)$$

Replacing expansion coefficients c_a by their moduli $|c_a|$, we introduce a new state $|\phi'\rangle$ as

$$|\phi'\rangle = \sum_{a=0}^{d-1} |c_a| |a\rangle = \sum_{b=0}^{d-1} \tilde{c}'_b |\tilde{b}\rangle, \quad (21)$$

where expansion coefficients of $|\phi'\rangle$ in the basis $\{|\tilde{b}\rangle\}$ are denoted by \tilde{c}'_b . We further define another state $|\phi''\rangle$ by replacing \tilde{c}'_b by $|\tilde{c}'_b|$, that is,

$$|\phi''\rangle = \sum_{b=0}^{d-1} |\tilde{c}'_b| |\tilde{b}\rangle. \quad (22)$$

Using Eqs. (7) and (8), we can readily show that the following relations hold:

$$\langle \phi' | P | \phi' \rangle \geq |\langle \phi | P | \phi \rangle|, \quad (23a)$$

$$\langle \phi' | Q | \phi' \rangle = \langle \phi | Q | \phi \rangle, \quad (23b)$$

and

$$\langle \phi'' | P | \phi'' \rangle = \langle \phi' | P | \phi' \rangle, \quad (24a)$$

$$\langle \phi'' | Q | \phi'' \rangle \geq |\langle \phi' | Q | \phi' \rangle|. \quad (24b)$$

Note that $\langle \phi'' | P | \phi'' \rangle$ and $\langle \phi'' | Q | \phi'' \rangle$ are real, and therefore, $\langle \phi'' | P | \phi'' \rangle = \langle \phi'' | P^\dagger | \phi'' \rangle$ and $\langle \phi'' | Q | \phi'' \rangle = \langle \phi'' | Q^\dagger | \phi'' \rangle$. Thus we have

$$\frac{|\langle \phi | Q | \phi \rangle| + |\langle \phi | P | \phi \rangle|}{2} \leq \langle \phi'' | \frac{P + P^\dagger + Q + Q^\dagger}{4} | \phi'' \rangle. \quad (25)$$

The right-hand side is clearly less than or equal to h , the greatest eigenvalue of H .

$$\langle \phi'' | \frac{P + P^\dagger + Q + Q^\dagger}{4} | \phi'' \rangle \leq h. \quad (26)$$

Combining this result and inequality (19), we obtain inequality (16).

B. Eigenstate of H with the greatest eigenvalue

Before proving statement (ii) of Theorem, we study the properties of the eigenstate of H with the greatest eigenvalue h . Some of them will be needed in the proof of statement (ii). We will show the following:

- (a) The greatest eigenvalue h is positive and not degenerate. The phase of corresponding eigenstate $|\Gamma\rangle$ can be chosen such that $\langle a|\Gamma\rangle$ is real and strictly positive for all a .
- (b) The eigenstate $|\Gamma\rangle$ is invariant under the Fourier transformation; $F|\Gamma\rangle = |\Gamma\rangle$, where F is the Fourier transform operator defined by

$$F = \sum_{a=0}^{d-1} |\tilde{a}\rangle \langle a|, \quad (27)$$

and, hence $\langle a|\Gamma\rangle = \langle \tilde{a}|\Gamma\rangle = \langle -a|\Gamma\rangle$.

- (c) The following relations hold:

$$\begin{aligned} h &= \langle \Gamma|Q|\Gamma\rangle = \langle \Gamma|Q^\dagger|\Gamma\rangle \\ &= \langle \Gamma|P|\Gamma\rangle = \langle \Gamma|P^\dagger|\Gamma\rangle. \end{aligned} \quad (28)$$

To show that the above statement (a) holds, some known properties of elementwise positive matrices will be employed. Here, we treat operators in the matrix representation based on the basis $\{|a\rangle\}_{a=0}^{d-1}$. We introduce an Hermitian matrix $H' \equiv H + \kappa \mathbf{1}$ with κ a real number. Off-diagonal part of H' is given by $(P + P^\dagger)/4$, all of whose elements are real and nonnegative. Diagonal part, $(Q + Q^\dagger)/4 + \kappa \mathbf{1}$, is denoted D , and all of whose diagonal elements are real and strictly positive for a sufficiently large κ . Now consider H'^d and expand it in terms of P , P^\dagger , and D . For any $i \leq j$, there is a term of the form $(P^\dagger/4)^{j-i} D^{d-j+i}$ that has a strictly positive (i, j) entry while other terms are elementwise nonnegative. For the (j, i) entry, a similar argument can be applied. Thus the matrix H'^d is elementwise strictly positive.

Now recall that, according to the Perron-Frobenius theorem, the eigenvalue of the largest modulus of a elementwise strictly positive matrix is real and non-degenerate, and the corresponding eigenvector can be chosen to have strictly positive components (see *e.g.* [10]). The eigenvalues of H'^d are clearly given by $(\kappa + \lambda_i)^d$ with λ_i being eigenvalues of H . Thus we conclude that the greatest eigenvalue of H is not degenerate and the corresponding eigenstate $|\Gamma\rangle$ can be chosen so that $\langle a|\Gamma\rangle > 0$ for all a .

To show that $h > 0$, note that the trace of H is 0. In the case of $d > 1$, this is possible only when $h > 0$ since h is the greatest eigenvalue and not degenerate. When $d = 1$, it is evident that $h = 1$.

Now we show that $F|\Gamma\rangle = |\Gamma\rangle$. It is easy to show that $FQF^\dagger = P^\dagger$ and $FPF^\dagger = Q$, and hence H commutes with F . This implies that $|\Gamma\rangle$ is an eigenstate of F since

the greatest eigenvalue h is not degenerate. The possible eigenvalues of F are 1, -1 , i , and $-i$. This is because $F^2 = T$, where T is the reflection operator given by

$$T = \sum_{a=0}^{d-1} |-a\rangle \langle a|, \quad (29)$$

and T satisfies $T^2 = \mathbf{1}$. Assume that $F|\Gamma\rangle = f|\Gamma\rangle$ with f being 1, -1 , i , or $-i$. This is explicitly written as

$$\sum_{a'=0}^{d-1} \langle a|F|a'\rangle \langle a'|\Gamma\rangle = f \langle a|\Gamma\rangle, \quad (30)$$

where $\langle a|F|a'\rangle = \omega^{aa'}/\sqrt{d}$. Setting $a = 0$, we observe

$$\frac{1}{\sqrt{d}} \sum_{a'=0}^{d-1} \langle a'|\Gamma\rangle = f \langle 0|\Gamma\rangle. \quad (31)$$

This requires that $f = 1$ since $\langle a|\Gamma\rangle > 0$ for all a . From $F|\Gamma\rangle = |\Gamma\rangle$, it immediately follows that $\langle a|\Gamma\rangle = \langle \tilde{a}|\Gamma\rangle = \langle -a|\Gamma\rangle$.

Further, the invariance $F|\Gamma\rangle = |\Gamma\rangle$ implies

$$\langle \Gamma|Q|\Gamma\rangle = \langle \Gamma|F^\dagger Q F|\Gamma\rangle = \langle \Gamma|P|\Gamma\rangle. \quad (32)$$

Since $\langle a|\Gamma\rangle$ and $\langle \tilde{b}|\Gamma\rangle$ are real, $\langle \Gamma|P|\Gamma\rangle$ and $\langle \Gamma|Q|\Gamma\rangle$ are also real. We therefore find

$$\langle \Gamma|Q|\Gamma\rangle = \langle \Gamma|Q^\dagger|\Gamma\rangle = \langle \Gamma|P|\Gamma\rangle = \langle \Gamma|P^\dagger|\Gamma\rangle, \quad (33)$$

which shows that each one is equal to h . Thus we obtain Eq. (28).

Explicit analytical solutions of h and $|\Gamma\rangle$ in general dimension d have not been obtained, but we collect some of the results in the low dimensional cases in Appendix.

C. Proof of statement (ii) in Theorem

“If part” is evident. When $|\phi\rangle = P^\alpha Q^\beta |\Gamma\rangle$, we find that

$$|\langle \phi|P|\phi\rangle| = |\langle \Gamma|P|\Gamma\rangle| = h, \quad (34)$$

$$|\langle \phi|Q|\phi\rangle| = |\langle \Gamma|Q|\Gamma\rangle| = h, \quad (35)$$

which shows that $|\phi\rangle$ satisfies equality in (16).

Proving “only if part” is rather involved. Suppose that $|\phi\rangle$ satisfies equality in (16). In the same way as in the proof of statement (i), we define $|\phi'\rangle$ and $|\phi''\rangle$ as follows:

$$|\phi\rangle = \sum_{a=0}^{d-1} c_a |a\rangle, \quad (36)$$

$$|\phi'\rangle = \sum_{a=0}^{d-1} |c_a| |a\rangle = \sum_{b=0}^{d-1} \tilde{c}'_b |\tilde{b}\rangle, \quad (37)$$

$$|\phi''\rangle = \sum_{b=0}^{d-1} |\tilde{c}'_b| |\tilde{b}\rangle. \quad (38)$$

This time equality should hold in all inequalities in the proof of statement (i).

First we note that equality in (26) is satisfied only if $|\phi''\rangle = |\Gamma\rangle$ up to a global phase since the greatest eigenvalue h is not degenerate.

Second we examine equality in (24b), $\langle\phi''|Q|\phi''\rangle = |\langle\phi'|Q|\phi'\rangle|$. This is explicitly written as

$$\sum_{b=0}^{d-1} |\tilde{c}'_{b+1}| |\tilde{c}'_b| = \left| \sum_{b=0}^{d-1} \tilde{c}'_{b+1}{}^* \tilde{c}'_b \right|, \quad (39)$$

which implies that all terms on the right-hand side must have the same phase factor, that is, $\tilde{c}'_{b+1}{}^* \tilde{c}'_b = |\tilde{c}'_{b+1} \tilde{c}'_b| u$ with u being a complex number of unit modulus. This relation can be rewritten as

$$\frac{\tilde{c}'_b}{|\tilde{c}'_b|} = u \frac{\tilde{c}'_{b+1}}{|\tilde{c}'_{b+1}|}, \quad (b = 0, 1, \dots, d-1), \quad (40)$$

Note that $|\tilde{c}'_b| > 0$ for all b since $|\phi''\rangle = |\Gamma\rangle$, and the above relation is well defined. Using this relation successively we obtain

$$\frac{\tilde{c}'_b}{|\tilde{c}'_b|} = u^{-b} \frac{\tilde{c}'_0}{|\tilde{c}'_0|}. \quad (41)$$

Setting $b = d$ and remembering $\tilde{c}'_d = \tilde{c}'_0$ by our convention, we find that the phase factor u must be a d th root of unity, ω^α with some integer α . Thus the b dependence of the phase of \tilde{c}'_b is given by $\omega^{-\alpha b}$, from which we conclude that $|\phi'\rangle = P^\alpha |\phi''\rangle = P^\alpha |\Gamma\rangle$ up to a global phase.

Let us now turn to the equality in (23a), $\langle\phi'|P|\phi'\rangle = |\langle\phi|P|\phi\rangle|$, which is explicitly written as

$$\sum_{a=0}^{d-1} |c_{a+1}| |c_a| = \left| \sum_{a=0}^{d-1} c_{a+1}^* c_a \right|, \quad (42)$$

Since $|\phi'\rangle = P^\alpha |\Gamma\rangle$, we have $|c_a| > 0$ for all a . We can repeat a similar argument to the preceding one, and we find that $|\phi\rangle$ is given by $Q^\beta |\phi'\rangle$ with some integer β . Combining this and the previous result, $|\phi'\rangle = P^\alpha |\Gamma\rangle$, we finally conclude that $|\phi\rangle = P^\alpha Q^\beta |\Gamma\rangle$ up to a global phase.

IV. CONTINUUM LIMIT

It is well known that, in the continuous quantum mechanics, the minimum uncertainty states are given by coherent states. The coherent states are eigenstates of the annihilation operator, and are given by translationally shifting the ground state of a harmonic oscillator in the phase space. In this section we will show that the minimum uncertainty state $|\Gamma\rangle$ obtained in the preceding section approaches the ground state of a harmonic oscillator as the dimension d goes to infinity.

We start by writing the eigen equation $H|\phi\rangle = \lambda|\phi\rangle$ in the position basis $\{|a\rangle\}$.

$$\frac{1}{4} \left(c_{a+1} + c_{a-1} + 2 \cos \left(\frac{2\pi}{d} a \right) c_a \right) = \lambda c_a, \quad (43)$$

where $c_a = \langle a|\phi\rangle$. Dickinson and Steiglitz [11] realized that this equation (43) is a discrete version of the Mathieu equation by identifying $c_{a+1} - 2c_a + c_{a-1}$ with the central second difference. To extend this idea further, we consider the following limit: By introducing the lattice constant ϵ , we define the system size $L = \epsilon d$. The system size L and the dimension d go to infinity, and the lattice constant ϵ goes to zero, while $\sigma \equiv \sqrt{\epsilon L/(2\pi)}$ is fixed. It is this σ that determines the scale of length. The factor 2π in the definition of σ is just for later convenience. The position variable x is defined by $x = a\epsilon$. Here the range of the discrete position index a is taken to be $[-(d-1)/2] \leq a \leq [(d-1)/2]$ where the symbol $[\cdot]$ means the floor function. This ensures that, in the large d limit, x becomes a continuous variable ranging from $-\infty$ to $+\infty$. Note that in this scheme we have

$$O(\epsilon^2) = O(1/L^2) = O(1/d). \quad (44)$$

Now we rewrite Eq. (43) as

$$-\frac{1}{2} \frac{\delta^2 c_a}{\epsilon^2} + \frac{2}{\epsilon^2} \sin^2 \left(\frac{\pi}{d} a \right) c_a = \frac{2}{\epsilon^2} (1 - \lambda) c_a, \quad (45)$$

where δ^2 is the central second difference given by

$$\delta^2 c_a = c_{a+1} - 2c_a + c_{a-1}. \quad (46)$$

By introducing the wave function $\phi(x) = c_a \sqrt{\epsilon}$, we observe

$$\frac{\delta^2 c_a}{\epsilon^2} \sqrt{\epsilon} = \phi''(x) + O\left(\frac{1}{d}\right), \quad (47)$$

and

$$\frac{2}{\epsilon^2} \sin^2 \left(\frac{\pi}{d} a \right) = \frac{x^2}{2\sigma^4} + O\left(\frac{1}{d}\right). \quad (48)$$

Thus, in the leading order, Eq. (45) takes the form

$$-\frac{1}{2} \phi''(x) + \frac{x^2}{2\sigma^4} \phi(x) = \frac{2}{\epsilon^2} (1 - \lambda) \phi(x), \quad (49)$$

which is the Schroedinger equation of the harmonic oscillator with the angular frequency given by $1/\sigma^2$. The eigen energy of this harmonic oscillator is given by $(n + 1/2)/\sigma^2$ where $n = 0, 1, \dots$. We thus find

$$\lambda = 1 - \left(n + \frac{1}{2} \right) \frac{\pi}{d}, \quad (50)$$

from which we obtain the asymptotic expression of the greatest eigenvalue h to be

$$h = 1 - \frac{\pi}{2d}, \quad (\text{as } d \rightarrow \infty). \quad (51)$$

The corresponding ground state wave function is given by a Gaussian function

$$\phi(x) \propto \exp \left(-\frac{x^2}{2\sigma^2} \right) = \exp \left(-\frac{\pi}{d} a^2 \right). \quad (52)$$

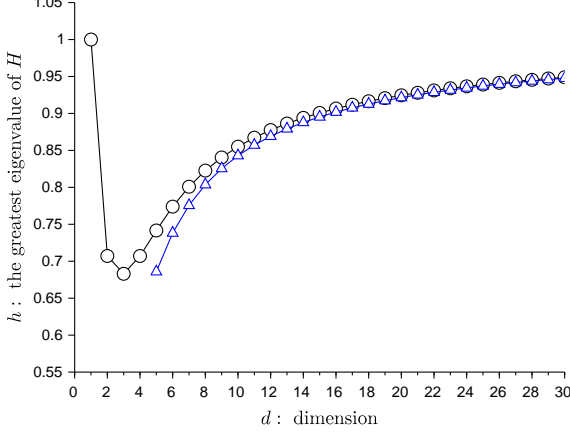


FIG. 2. (Color online) The greatest eigenvalue h of the operator H vs dimension d . The circles represent the exact values calculated by diagonalizing H analytically or numerically. The values of the asymptotic formula Eq. (51) are plotted by triangles.

Thus the asymptotic form of the minimum uncertainty state $|\Gamma\rangle$ is given by

$$\langle a|\Gamma\rangle = \mathcal{N} \exp\left(-\frac{\pi}{d}a^2\right), \quad (\text{as } d \rightarrow \infty), \quad (53)$$

where $[-(d-1)/2] \leq a \leq [(d-1)/2]$, and \mathcal{N} is a normalization constant.

In Fig. 2 we compare the exact values of h with those obtained by the asymptotic formula Eq. (51). This shows that the asymptotic form is already a rather good approximation for relatively low dimensions. The components of the minimum uncertainty state $\langle a|\Gamma\rangle$, the values by numerical calculation and by the asymptotic form Eq. (53), are plotted in Fig. 3. We see that the asymptotic form provides an unexpectedly good approximation even for the $d = 5$ case.

We briefly sketch how the inequality (16) of the certainty $C(\phi)$ is reduced to the usual uncertainty relation of the position and momentum variables in the continuum limit. First we analyze the expectation value $\langle \phi|Q|\phi\rangle$. In the continuum limit, the summation over a becomes an integral over x , and the exponential function $\exp(i\frac{2\pi}{L}a) = \exp(i\frac{2\pi}{L}x)$ can be expanded. Thus we have

$$\begin{aligned} \langle \phi|Q|\phi\rangle &= \sum_a e^{i\frac{2\pi}{L}a} |c_a|^2 \\ &= 1 + i\frac{2\pi}{L} \langle \hat{x} \rangle - \frac{1}{2} \left(\frac{2\pi}{L}\right)^2 \langle \hat{x}^2 \rangle + O\left(\frac{1}{d^{3/2}}\right), \end{aligned} \quad (54)$$

where

$$\begin{aligned} \langle \hat{x} \rangle &= \int dx \phi^*(x) x \phi(x), \\ \langle \hat{x}^2 \rangle &= \int dx \phi^*(x) x^2 \phi(x). \end{aligned}$$

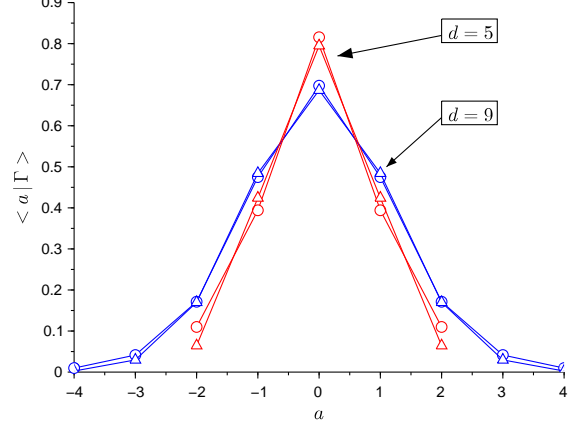


FIG. 3. (Color online) The components of the minimum uncertainty state $|\Gamma\rangle$. The components in the position basis $\{|a\rangle, -(d-1)/2 \leq a \leq (d-1)/2\}$ are plotted vs a for $d = 5$ and $d = 9$ cases. The circles are the values obtained by numerical calculations. The triangles represent the values by the asymptotic form of Eq. (53). The normalization constants \mathcal{N} are determined numerically.

The modulus of $\langle \phi|Q|\phi\rangle$ then takes the form

$$|\langle \phi|Q|\phi\rangle| = 1 - \frac{\pi}{\sigma^2 d} (\Delta x)^2 + O\left(\frac{1}{d^{3/2}}\right), \quad (55)$$

in terms of the standard deviation of position coordinate defined by $\Delta x = \sqrt{\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2}$. Similarly, $|\langle \phi|P|\phi\rangle|$ is expressed as

$$|\langle \phi|P|\phi\rangle| = 1 - \frac{\pi \sigma^2}{d} (\Delta p)^2 + O\left(\frac{1}{d^{3/2}}\right), \quad (56)$$

where Δp is the usual standard deviation of momentum coordinate. Meanwhile, the asymptotic form of h has already been obtained in Eq. (51). Combining all these results, we find that the inequality (16) of the certainty $C(\phi)$ is reduced to

$$\frac{1}{\sigma^2} (\Delta x)^2 + \sigma^2 (\Delta p)^2 \geq 1, \quad (57)$$

in the leading order of $1/d$.

It is evident that, for a given wave function $\phi(x)$, the scale factor σ is arbitrary, since σ is a sort of artifact in the procedure of the continuum-limit scheme. The left-hand side of the above inequality (57) takes the minimum value $2\Delta x \Delta p$ when $\sigma = \sqrt{\Delta x / \Delta p}$. Thus we arrive at the usual uncertainty relation of the position and momentum variables.

$$\Delta x \Delta p \geq \frac{1}{2}. \quad (58)$$

V. NON-NEGATIVE QUASI PROBABILITY

The minimum uncertainty states are defined as

$$|\alpha, \beta\rangle = P^\alpha Q^\beta |\Gamma\rangle, \quad (\alpha, \beta = 0, 1, \dots, d-1). \quad (59)$$

The position and momentum distributions of $|\alpha, \beta\rangle$ are given by

$$|\langle a|\alpha, \beta\rangle|^2 = \Gamma_{a-\alpha}^2, \quad (60)$$

$$|\langle \tilde{b}|\alpha, \beta\rangle|^2 = \Gamma_{b-\beta}^2. \quad (61)$$

Note that $\Gamma_a \equiv \langle a|\Gamma\rangle$ has a peak at $a = 0$, which can be seen from the analytical results in the low dimensional cases (see Appendix) and the numerical results for higher dimensions. Therefore, the position and momentum distribution of $|\alpha, \beta\rangle$ have a peak at $a = \alpha$ and $b = \beta$, respectively.

The d^2 minimum uncertainty states $|\alpha, \beta\rangle$ are not mutually orthogonal, but they comprise an overcomplete set in the state vector space \mathbb{C}^d . The completeness relation of $|\alpha, \beta\rangle$ takes the form

$$\frac{1}{d} \sum_{\alpha, \beta=0}^{d-1} |\alpha, \beta\rangle \langle \alpha, \beta| = \mathbf{1}. \quad (62)$$

In order to derive this completeness relation, we employ the following useful identity which holds for any operator Ω :

$$\sum_{\alpha, \beta=0}^{d-1} \omega^{\alpha b - \beta a} P^\alpha Q^\beta \Omega Q^{-\beta} P^{-\alpha} = d \operatorname{tr} [Q^{-b} P^{-a} \Omega] P^a Q^b, \quad (63)$$

or equivalently,

$$\begin{aligned} & P^\alpha Q^\beta \Omega Q^{-\beta} P^{-\alpha} \\ &= \frac{1}{d} \sum_{a, b=0}^{d-1} \omega^{-\alpha b + \beta a} \operatorname{tr} [Q^{-b} P^{-a} \Omega] P^a Q^b. \end{aligned} \quad (64)$$

This identity can be obtained by using the commutation relation $QP = \omega PQ$ together with the mutual orthogonality and completeness of the set of operators $\{P^\alpha Q^\beta\}_{\alpha, \beta=0}^{d-1}$ in the operator space. Setting $a = b = 0$ and $\Omega = |\Gamma\rangle \langle \Gamma|$ in the above identity (63), we obtain the completeness of the minimum uncertain states (62).

With these observations, it is reasonable to define the following quasi probability distribution $D(\alpha, \beta)$ for a given state ρ with respect to the position and momentum coordinates α and β :

$$D(\alpha, \beta) \equiv \frac{1}{d} \langle \alpha, \beta | \rho | \alpha, \beta \rangle = \operatorname{tr} [\rho \Delta(\alpha, \beta)], \quad (65)$$

where we introduced the phase point operator $\Delta(\alpha, \beta)$ given by

$$\Delta(\alpha, \beta) = \frac{1}{d} |\alpha, \beta\rangle \langle \alpha, \beta|. \quad (66)$$

Note that $D(\alpha, \beta)$ is non-negative and normalized to be unity when summed over all phase space points (α, β) . However, the states $|\alpha, \beta\rangle$ are not mutually orthogonal, and therefore distinct phase space points (α, β) are not regarded as exclusive events. This is the reason why we call $D(\alpha, \beta)$ a *quasi* probability distribution.

The phase point operator $\Delta(\alpha, \beta)$ satisfies the following relations if summed over α or β :

$$\sum_{\beta=0}^{d-1} \Delta(\alpha, \beta) = \sum_{a=0}^{d-1} \Gamma_{a-\alpha}^2 |a\rangle \langle a|, \quad (67)$$

$$\sum_{\alpha=0}^{d-1} \Delta(\alpha, \beta) = \sum_{b=0}^{d-1} \Gamma_{b-\beta}^2 |\tilde{b}\rangle \langle \tilde{b}|. \quad (68)$$

The first equation (67) can be obtained by summing over β in Eq. (64) with $\Omega = |\Gamma\rangle \langle \Gamma|$. Similarly the second equation (68) also follows from Eq. (64). These relations (67) and (68) imply that the quasi probability distribution $D(\alpha, \beta)$ has the following marginal distributions:

$$\sum_{\beta=0}^{d-1} D(\alpha, \beta) = \sum_{a=0}^{d-1} \Gamma_{a-\alpha}^2 \langle a | \rho | a \rangle, \quad (69)$$

$$\sum_{\alpha=0}^{d-1} D(\alpha, \beta) = \sum_{b=0}^{d-1} \Gamma_{b-\beta}^2 \langle \tilde{b} | \rho | \tilde{b} \rangle. \quad (70)$$

We find that the marginal distributions are smeared out in the sense that $D(\alpha, \beta)$ summed over β for example gives the weighted average of $\langle a | \rho | a \rangle$ with the weight centered at $a = \alpha$.

It is evident that the phase point operator $\Delta(\alpha, \beta)$ respects the translational covariance,

$$P^a Q^b \Delta(\alpha, \beta) Q^{-b} P^{-a} = \Delta(\alpha + a, \beta + b), \quad (71)$$

which implies that if $D(\alpha, \beta)$ is the quasi probabilities of a state ρ then the quasi probabilities of $\rho' = P^a Q^b \rho Q^{-b} P^{-a}$ is given by $D(\alpha - a, \beta - b)$. The phase point operator is also covariant under the Fourier transformation; that is, $F \Delta(\alpha, \beta) F^\dagger = \Delta(-\beta, \alpha)$, but not covariant under the more general symplectic transformation considered in [12–14].

It is desirable that the quasi probability distribution completely determines the state of the system. This requires that the set of phase point operators $\{\Delta(\alpha, \beta)\}_{\alpha, \beta=0}^{d-1}$ should be complete in the operator space. To see this we calculate the Fourier transform of $\Delta(\alpha, \beta)$.

$$\begin{aligned} \tilde{\Delta}(m, n) &\equiv \frac{1}{d} \sum_{\alpha, \beta=0}^{d-1} \omega^{\alpha n - \beta m} \Delta(\alpha, \beta) \\ &= \frac{1}{d} \langle \Gamma | Q^{-n} P^{-m} | \Gamma \rangle P^m Q^n \\ &= \frac{1}{d} \langle \Gamma | Q^n P^m | \Gamma \rangle P^m Q^n. \end{aligned} \quad (72)$$

We employed Eq. (63) with $\Omega = |\Gamma\rangle \langle \Gamma|$ to obtain the second line of the above equation, and the reflection symmetry $T |\Gamma\rangle = |\Gamma\rangle$ was also used in the last line. Since

the set of operators $\{P^m Q^n\}_{m,n=0}^{d-1}$ is complete, the completeness of the phase point operators is equivalent to the conditions,

$$\langle \Gamma | Q^n P^m | \Gamma \rangle \neq 0, \quad (m, n = 0, 1, \dots, d-1). \quad (73)$$

Here we have different results depending on whether the dimension d is even or odd. When d is odd, we believe that the set of phase point operators $\Delta(\alpha, \beta)$ is complete. This is because our numerical calculation shows that the conditions (73) are satisfied at least up to $d = 25$, although we can not yet prove it analytically except for low dimensional cases ($d=3,5$) in which $|\Gamma\rangle$ is known analytically.

When d is even, on the other hand, some of the conditions (73) are violated. For example, we find that

$$\langle \Gamma | Q^{d/2} P^{d/2} | \Gamma \rangle = \frac{1}{2h} \langle \Gamma | \{Q^{d/2} P^{d/2}, H\} | \Gamma \rangle = 0, \quad (74)$$

since the operator $Q^{d/2} P^{d/2}$ anticommutes with H . We can also show that

$$\begin{aligned} \langle \Gamma | Q^{d/2} P^m | \Gamma \rangle &= 0, \quad (m = \text{odd}), \\ \langle \Gamma | Q^n P^{d/2} | \Gamma \rangle &= 0, \quad (n = \text{odd}). \end{aligned} \quad (75)$$

Thus the phase point operators $\Delta(\alpha, \beta)$ are not complete if d is even. Let us examine the qubit ($d = 2$) case more closely. In this case we can write the phase point operator as

$$\Delta(\alpha, \beta) = \frac{1}{4} \left(1 + \mathbf{n}^{(\alpha, \beta)} \cdot \boldsymbol{\sigma} \right), \quad (\alpha, \beta = 0, 1) \quad (76)$$

where the Bloch vectors $\mathbf{n}^{(\alpha, \beta)}$ are given in Eq. (13). Since the y -components of $\mathbf{n}^{(\alpha, \beta)}$ are 0, the set of $\Delta(\alpha, \beta)$'s is not complete in the whole qubit space. However, it is interesting that it is still complete in the qubit space of real amplitudes.

For the odd dimensional system, the Wigner function of Wootters [15] and Cohendet *et al.* [16] is defined as $D_W(\alpha, \beta) = \text{tr}[\rho \Delta_W(\alpha, \beta)]$ with the phase point operator given by

$$\Delta_W(\alpha, \beta) = \frac{1}{d} P^\alpha Q^\beta T Q^{-\beta} P^{-\alpha}. \quad (77)$$

This Wigner function has sharp marginal distributions in the sense that

$$\sum_{\beta=0}^{d-1} \Delta_W(\alpha, \beta) = |\alpha\rangle \langle \alpha|, \quad (78)$$

$$\sum_{\alpha=0}^{d-1} \Delta_W(\alpha, \beta) = |\tilde{\beta}\rangle \langle \tilde{\beta}|. \quad (79)$$

However, the Wigner function $D_W(\alpha, \beta)$ may take negative values and non-negative only for special states called

stabilizer states [13], since $\Delta_W(\alpha, \beta)$ is not positive semi-definite. Using the mutual orthogonality and completeness of $\Delta_W(\alpha, \beta)$ in the operator space, we can easily express $\Delta(\alpha, \beta)$ in terms of $\Delta_W(\alpha, \beta)$. The result is given by

$$\Delta(\alpha, \beta) = \sum_{\alpha', \beta'=0}^{d-1} w(\alpha - \alpha', \beta - \beta') \Delta_W(\alpha', \beta'), \quad (80)$$

where

$$w(\alpha, \beta) = \langle \Gamma | \Delta_W(\alpha, \beta) | \Gamma \rangle. \quad (81)$$

We see that the phase point operator $\Delta(\alpha, \beta)$ built with the minimum uncertainty states can be written in the form of convolution of the weight $w(\alpha, \beta)$ and $\Delta_W(\alpha, \beta)$, and thus it acquires non-negativity at the cost of losing the sharp marginal property.

VI. DISCUSSION

The quasi probability distribution based on the minimum uncertainty states is non-negative, but its marginal distributions are smeared out as shown in Eqs. (69,70). A natural question is whether there exists a non-negative quasi probability distribution which satisfies sharper marginal conditions. In what follows, we show the answer is “no” as long as the translational covariance is assumed.

Let $\Lambda(\alpha, \beta)$ be phase point operators of a non-negative quasi probability distribution with the translational covariance. To quantify the sharpness of the marginal distributions, we define

$$\sigma \equiv \left| \text{tr} \left[\sum_{\beta=0}^{d-1} \Lambda(\alpha, \beta) Q \right] \right|, \quad (82)$$

$$\tau \equiv \left| \text{tr} \left[\sum_{\alpha=0}^{d-1} \Lambda(\alpha, \beta) P \right] \right|. \quad (83)$$

Because of the translational covariance, σ and τ are independent of α and β , respectively. In the case of $\Delta_W(\alpha, \beta)$ by Wootters and Cohendet *et al.*, we find that $\sigma = \tau = 1$ since the marginal conditions are perfectly sharp as shown in Eqs. (78,79). However, for $\Delta(\alpha, \beta)$ based on the minimum uncertainty states, we have $\sigma = \tau = h$, which is less than 1 if $d \geq 2$.

The translational covariance implies that $\Lambda(\alpha, \beta)$ can be written as

$$\Lambda(\alpha, \beta) = \frac{1}{d} P^\alpha Q^\beta K Q^{-\beta} P^{-\alpha}, \quad (84)$$

where $K = d\Lambda(0, 0)$ is a Hermitian operator with $\text{tr} K = 1$ since $\Lambda(\alpha, \beta)$ should be Hermitian and normalized as $\sum_{\alpha, \beta=0}^{d-1} \Lambda(\alpha, \beta) = 1$. In addition, K should be positive semi-definite to ensure that the quasi probabilities are

non-negative. Thus K can be regarded as a state on \mathbb{C}^d . In terms of K , the measures of sharpness, σ and τ , take the following simple form:

$$\sigma = |\text{tr}[KQ]|, \quad \tau = |\text{tr}[KP]|. \quad (85)$$

Here it should be noticed that the theorem proved in Sec. III holds also for mixed states; that is, for any state ρ , we have

$$|\text{tr}[\rho Q] \text{tr}[\rho P]| \leq h^2, \quad (86)$$

where equality holds if and only if $\rho = |\alpha, \beta\rangle\langle\alpha, \beta|$. This can be shown by the following inequalities:

$$\begin{aligned} |\text{tr}[\rho Q] \text{tr}[\rho P]|^{1/2} &\leq \frac{1}{2} (|\text{tr}[\rho Q]| + |\text{tr}[\rho P]|) \\ &\leq \sum_i \lambda_i \frac{1}{2} (|\langle\phi_i|Q|\phi_i\rangle| + |\langle\phi_i|P|\phi_i\rangle|) \leq h, \end{aligned} \quad (87)$$

where we used the spectral decomposition of ρ ; $\rho = \sum_i \lambda_i |\phi_i\rangle\langle\phi_i|$.

Using this extended theorem, we obtain

$$\sigma\tau \leq h^2, \quad (88)$$

where equality holds if and only if $K = |\alpha_0, \beta_0\rangle\langle\alpha_0, \beta_0|$ with $\alpha_0, \beta_0 = 0, 1, \dots, d-1$. This implies that the upper bound of the sharpness $\sigma\tau$ is attained by $\Lambda(\alpha, \beta) = \Delta(\alpha + \alpha_0, \beta + \beta_0)$. Thus we conclude that the quasi probability distribution based on $\Delta(\alpha, \beta)$ is optimal and unique up to a cyclic relabeling of the position and momentum coordinates; $\alpha \rightarrow \alpha + \alpha_0$ and $\beta \rightarrow \beta + \beta_0$.

VII. SUMMARY AND CONCLUDING REMARKS

The aim of this paper is to construct the minimum uncertainty states and the non-negative quasi probability distribution for a qudit. They are the finite-dimensional counter parts of the coherent states and the Husimi function of the continuous quantum mechanics. We first defined the measure to quantify the uncertainty of each of the position and momentum coordinates. Then we showed that the product of these two measures satisfies a certain inequality. The minimum uncertainty states are the ones saturating this inequality. With them, we constructed the non-negative quasi probability distribution, of which marginal distributions are smeared out like in the case the Husimi function. This quasi probability distribution is optimal in the sense that there does not exist non-negative and translationally covariant quasi probability distribution with sharper marginal properties.

When d is even, our quasi probability distribution is not complete. It is well known that the Wigner function in the even d case is much more involved than in the odd d case (see *e.g.* [17, 18]). Further investigation for this even-odd issue of quasi probabilities is certainly needed.

The Wigner function may take negative values. In Refs. [16, 19], however, it is shown that one can define non-negative quasi probabilities by introducing an auxiliary variable into the Wigner function, and solve the dynamics of a quantum system stochastically. It will be of interest in future studies to apply our quasi probability distribution to this line of research.

Appendix: The greatest eigenvalue of H and its eigenstate

This appendix collects some analytical results of the greatest eigenvalue h of the operator H and its eigenstate $|\Gamma\rangle$ in some low-dimensional cases. The case of $d = 1$ is trivial; namely, h is equal to 1. For $d \geq 2$, some symmetry consideration is helpful. As we have shown in Sec. III B the state $|\Gamma\rangle$ is invariant under the Fourier transformation. The multiplicity of eigenvalues of the Fourier operator F in general dimensions is well known, see for example [20]. The eigenvalue 1 of F is non-degenerate for $d = 1, 2$, and 3, and doubly degenerate for $d = 4, 5, 6$, and 7.

It is easy to verify that the vector whose components in the position basis are given by

$$\xi_a = 1 + \sqrt{d}\delta_{a,0}, \quad (A.1)$$

is an eigenvector of F with eigenvalue 1. Therefore, in the cases of $d = 2$ and 3, $\Gamma_a = \langle a|\Gamma\rangle$ is given by ξ_a up to a normalization constant. Thus we have

$$h = \frac{\langle\xi|H|\xi\rangle}{\langle\xi|\xi\rangle} = \frac{\sqrt{d}+2}{2(\sqrt{d}+1)} = \begin{cases} \frac{1}{\sqrt{2}} & (d=2) \\ \frac{\sqrt{3}+1}{4} & (d=3) \end{cases}, \quad (A.2)$$

and

$$\Gamma_a = \frac{1 + \sqrt{d}\delta_{a,0}}{\sqrt{2}\sqrt{d+1}}, \quad (d=2, 3). \quad (A.3)$$

We introduce another vector $|\eta\rangle$ whose components are given by

$$\eta_a = \omega^a + \omega^{-a} + \sqrt{d}(\delta_{a,1} + \delta_{a,d-1}), \quad (A.4)$$

which is also an eigenstate of F with eigenvalue of 1. Note that the vectors $|\xi\rangle$ and $|\eta\rangle$ are linearly independent if $d \geq 4$. Therefore, for the cases of $d = 4, 5, 6$, and 7, the task is reduced to a two-dimensional diagonalization problem in the space spanned by the two vectors $|\xi\rangle$ and $|\eta\rangle$. The state $|\Gamma\rangle$ is expanded as $|\Gamma\rangle = c_\xi |\xi\rangle + c_\eta |\eta\rangle$, and the coefficients c 's and the eigenvalue h are determined by the following generalized eigen equation:

$$\begin{bmatrix} \langle\xi|H|\xi\rangle & \langle\xi|H|\eta\rangle \\ \langle\eta|H|\xi\rangle & \langle\eta|H|\eta\rangle \end{bmatrix} \begin{bmatrix} c_\xi \\ c_\eta \end{bmatrix} = \lambda \begin{bmatrix} \langle\xi|\xi\rangle & \langle\xi|\eta\rangle \\ \langle\eta|\xi\rangle & \langle\eta|\eta\rangle \end{bmatrix} \begin{bmatrix} c_\xi \\ c_\eta \end{bmatrix}, \quad (A.5)$$

where the larger of the two eigenvalues λ should be taken as h . The resultant expressions of h and Γ_a as functions

of d are quite complicated. Here we list only some simple results for each dimension d , which may be useful for further study.

In the case of $d = 4$, we obtain the following simple result:

$$h = \frac{1}{\sqrt{2}}, \quad (\text{A.6}) \quad \text{and}$$

and

$$\Gamma_a = \frac{1}{2\sqrt{2}} \left(\sqrt{2} + 1, 1, \sqrt{2} - 1, 1 \right). \quad (\text{A.7})$$

For the cases of $d = 5, 6$, we present only the results of h .

$$h = \frac{\sqrt{2\sqrt{5} + 70} + \sqrt{5} + 1}{16}, \quad (d = 5), \quad (\text{A.8})$$

$$h = \frac{\sqrt{14} + \sqrt{6}}{8}, \quad (d = 6). \quad (\text{A.9})$$

We used $\cos(2\pi/5) = (\sqrt{5} - 1)/4$ in the $d = 5$ case.

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